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XXVI Cycle

Fast multikernel algorithms for low frequency and related operators

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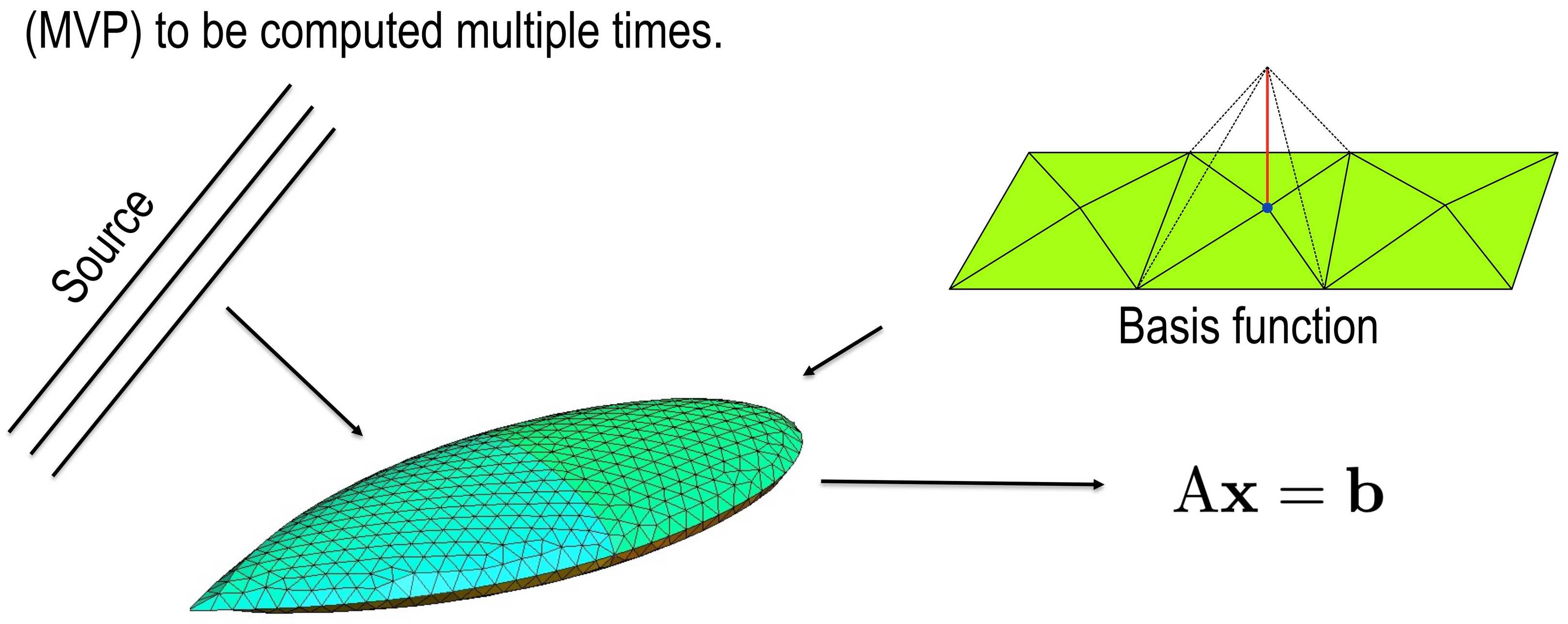
Research context and motivation

- The boundary element method (BEM) is a widespread numerical computational method for problems that can be formulated as integral equations from partial differential equations (PDE). In particular, it can be used in PDEs where is defined the green's function, such as the Helmotz or the Laplace equation.
- In a scattering problem, the goal is to find the scattering function across a determined domain. The object surface is discretized with a mesh. Then, the basis functions are defined aiming to approximate the unknown function. The boundary conditions are determined by an an external excitation source interacting with the object.
- The testing procedure leads then to a matrix linear system in the form Ax = b. The linear system is often solved with iterative solvers, which require the matrix vector product (MVP) to be computed multiple times.



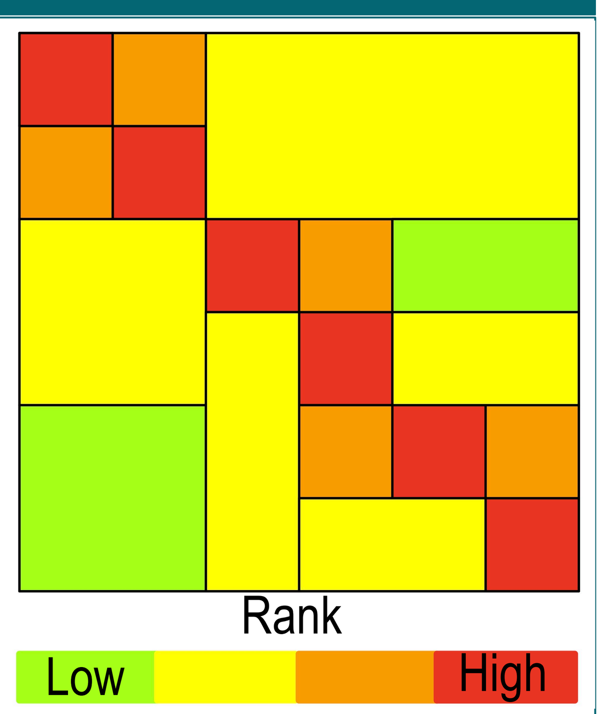
- Each element of the matrix is a double integration involving a kernel function across the supports of the corresponding basis functions. This integration can be approximated with different quadrature schemes. For non-singular interactions, it can be written as:

$$\mathrm{A}_{ij} = \int_{\mathrm{supp}(f_i)} f_i(\mathbf{r}) \int_{\mathrm{supp}(f_j)} f_j(\mathbf{r}')\gamma(\mathbf{r},\mathbf{r}')d\mathbf{r}'d\mathbf{r} = \sum_{p=1}^{N_{int}}\sum_{q=1}^{N_{int}} w_p w_q c_{i,p} c_{j,q} \gamma(\mathbf{r}_{i,p},\mathbf{r}'_{j,q})$$

where $\gamma(\mathbf{r},\mathbf{r}')$ is a kernel function that depends on the discretized operator.

Addressed research questions/problems

- Given a N x N matrix A, without any further improvements, the time complexity of the MVP would be O(N²). Nevertheless, even if the global rank is full, BEM matrices exhibit block-wise rank deficiency. Therefore, they can be divided hierarchically in blocks where a low rank decomposition can be computed, allowing a fast MVP.
- In our work, we are interested in accelerating the MVP of matrices in the context of static and low frequency operators. Primarily, we are focused on formulations requiring the discretization of multiple operators, such as:



1. **Indirect combined-field integral equation: Brakhage–Werner equation**
   This formulation allows to solve a scalar scattering problem described by the Helmotz equation of a sound-soft rough surface. The discretized formulation yields

$$\left(\frac{1}{2}\mathrm{I} + \mathrm{D} - j\eta\mathrm{S}\right)\mathbf{x} = \mathbf{b} \qquad \mathbf{u} = (\mathrm{D} - j\eta\mathrm{S})\,\mathbf{x}$$

   Where **b** is determined by the Dirichlet boundary conditions, and **u** is the scattered field.
2. **Symmetric formulation**
   The symmetric formulation of the electroencephalography (EEG) is one of the most common techniques used to solve the EEG forward problem thanks to the provided high accuracy. The discretized matrix contains multiple operators that are computed across the boundaries of the layers of the object scaled by the appropriate conductivities.

$$\begin{bmatrix} (\sigma_1+\sigma_2)\mathrm{N}_{11} & -2\mathrm{D}^*_{11} & -\sigma_2\mathrm{N}_{12} & \mathrm{D}^*_{12} & 0 & 0 & 0 \\ -2\mathrm{D}_{11} & (\sigma_1^{-1}+\sigma_2^{-1})\mathrm{S}_{11} & \mathrm{D}_{12} & \sigma_2^{-1}\mathrm{S}_{12} & 0 & 0 & 0 \\ -\sigma_2\mathrm{N}_{21} & \mathrm{D}^*_{21} & (\sigma_2+\sigma_3)\mathrm{N}_22 & -2\mathrm{D}^*_{22} & -\sigma_3\mathrm{N}_{23} & \mathrm{D}^*_{23} & 0 \\ \mathrm{D}_{21} & \sigma_2^{-1}\mathrm{S}_{21} & -2\mathrm{D}_{22} & (\sigma_2^{-1}+\sigma_3^{-1})\mathrm{S}_{22} & \mathrm{D}_{23} & -\sigma_3^{-1}\mathrm{S}_{23} & 0 \\ 0 & 0 & -\sigma_3\mathrm{N}_{32} & \mathrm{D}^*_{32} & (\sigma_3+\sigma_4)\mathrm{N}_{33} & -2\mathrm{D}^*_{33} & \cdots \\ 0 & 0 & \mathrm{D}_{32} & \sigma_3^{-1}\mathrm{S}_{32} & -2\mathrm{D}_{33} & (\sigma_3^{-1}+\sigma_4^{-1})\mathrm{S}_{33} & \cdots \\ 0 & 0 & 0 & 0 & \cdots & \cdots & \ddots \end{bmatrix}$$

Submitted and published works

Novel contributions

- In order to accelerate the MVPs of the described formulations, commonly it is required to compress every discretized operator independently.
- Our contribution is oriented toward the compression of multiple kernels at once by exploiting the factorization of the quadrature schemes.

Operator	γ
$\mathcal{S}$	$G(\mathbf{r},\mathbf{r}')$
$\mathcal{D}$	$\partial_{\mathbf{n}'}G(\mathbf{r}-\mathbf{r}')$
$\mathcal{D}^*$	$\partial_{\mathbf{n}}G(\mathbf{r}-\mathbf{r}')$
$\mathcal{N}$	$\partial_{\mathbf{n}',\mathbf{n}}G(\mathbf{r}-\mathbf{r}')$

$\mathrm{B}_{\mathcal{S}}$, $\mathrm{B}_{\mathcal{D}}$, $\mathrm{B}_{\mathcal{D}^*}$, $\mathrm{B}_{\mathcal{N}}$

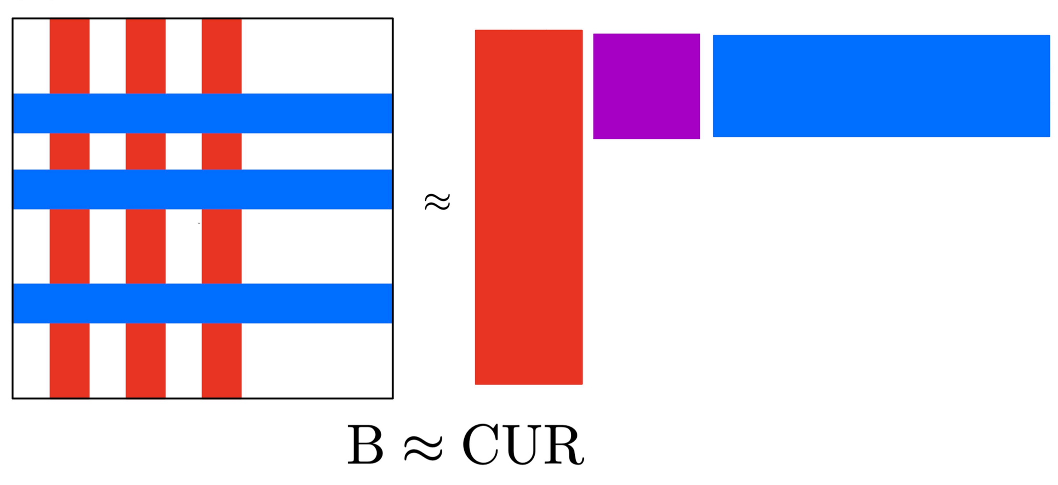
Classic method → 4 compressions

Novel method → 1 compression

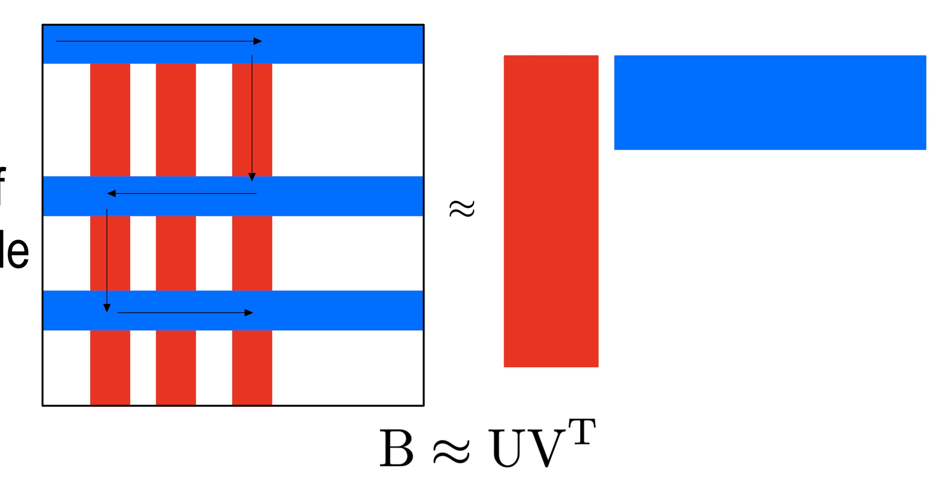
- This approach allows to accelerate the MVPs of these formulations achieving competitive performances and significantly less memory consumptions. As a consequence, this scheme is particularly suited for GPUs, where memories and bandwidths are limited.

Adopted methodologies

- Due to the nature of the green's function related kernels, we can determine which boxes yield a low rank interaction. Given a matrix block B, its low rank approximation can be computed with several algorithms such as
- **CUR/Pseudo-skeleton approximation**
  The matrix is approximated by sampling rows and columns and computing the inverse (or the pseudo-inverse) of the intersection. The challenge is to select an appropriate number of rows and columns to not exceed the predefined relative error.



- **Adaptive Cross Approximation (ACA)**
  The ACA is an heuristic algorithm that approximates a matrix with the product of two orthogonal matrices. The most notable version involves a pivoting strategy to select locally the best column and row at each iteration.



Future work

- The scheme is planned to be extended to high order methods and to other formulations.
- In addition, we are currently working on the sampling algorithm in order to select the minimum number of columns and rows without exceeding the predefined global error.

List of attended classes

- ESOA - Advanced Computational Electromagnetics (MCSA COMPETE) (2022, 30)
- ESOA - Microwave Imaging and Diagnostics: Theory, Techniques and Applications (2021, 24)
- ESOA - Advanced Mathematics for Antenna Analysis (2021, 30)
- 01MMRRV - Tecniche numeriche avanzate per l'analisi ed il progetto di antenne (2021, 20)
- 01UJDRV - Integral operators and fast solvers: a cross-disciplinary excursus on the best of FFT'companions (2022, 30)
- 02RBYKI - From science to business: how to get technology out of laboratories and into practical applications (2021, 20)
- 01DOBRV - Mathematical-physical theory of electromagnetism (2022, 15)
- 02LWHRV Communication (5, 2022)
- 01UIZRV Microwave sensing and imaging for innovative applications in health and food industry (2022, 20)
- 03QTIIU Mimetic learning (2021, 20)
- 08IXTRV Project management (2022, 5)
- 01RISRV Public speaking (2022, 5)