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XXXVII Cycle

A Refinement-Free Preconditioner for the Symmetric Formulation in EEG

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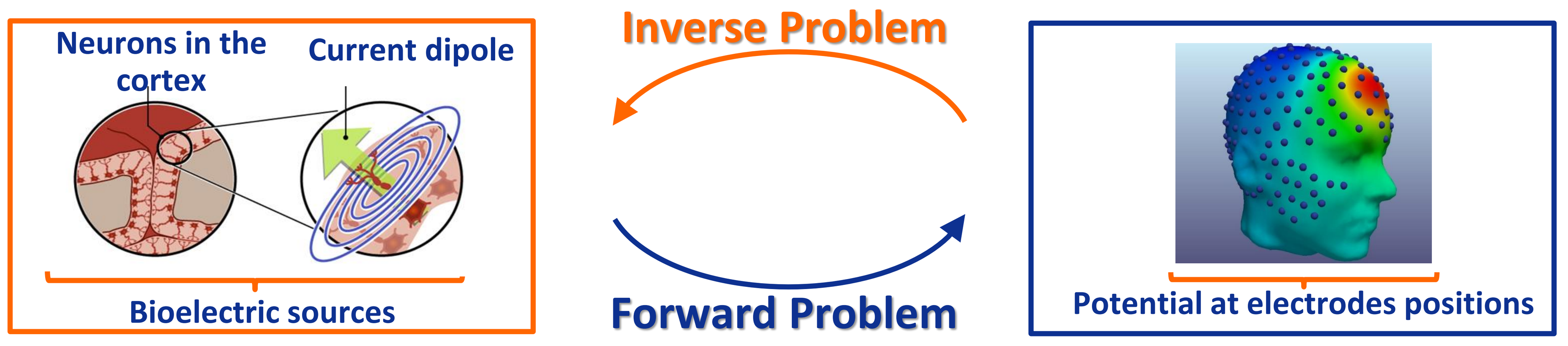
Research context and motivation

- **High resolution Electroencephalography** (**HR-EEG**) is a non-invasive, functional **neuroimaging technique**: it allows to determine position, strength and direction of the neural sources starting from the measure of the potential distribution at the scalp. HR-EEG is nowadays commonly used for the **pre-surgical epileptogenic source localization** in patients affected by focal epilepsy. Given the criticality of the applications, the source localization algorithms should provide **accurate** and **reliable** results **in short time**.
- EEG neuroimaging aims at solving an **inverse problem** (i.e. the identification of the neural sources from the measured scalp potential). It is commonly tackled by the iterative solution of the **forward EEG problem** (i.e. finding the scalp potential given a known source current inside the brain). Therefore, the accuracy and the resolution time of the forward problem affect the ones of the inverse problem. This work consists in the **stabilization** of a well-known integral formulation for the solution of the forward problem.



Addressed research questions/problems

- Under the **quasi-static** approximation, the problem is modeled by Poisson's equation, characterizing a **purely resistive model** of the head: $\nabla\cdot(\sigma\,\nabla V)=\nabla\cdot\boldsymbol{j}$.
- To recast this differential equation into a **boundary integral equation**, we model the biological tissues of the head, such as the brain, the skull and the scalp, as nested sets and assume **homogeneous conductivity** inside each. Moreover, we impose the continuity of the potential and of the current across the boundaries of these subdomains. We obtain the symmetric formulation:

$$\begin{cases}(\partial_{\boldsymbol{n}} v_{\Omega_{i+1}})_{\Gamma_i}-(\partial_{\boldsymbol{n}} v_{\Omega_i})_{\Gamma_i}=-\mathcal{D}^*_{i,i-1}p_{i-1}+2\mathcal{D}^*_{ii}p_i-\mathcal{D}^*_{i,i+1}p_{i+1}\\ \qquad +\sigma_i\mathcal{N}_{i,i-1}V_{i-1}-(\sigma_i+\sigma_{i+1})\mathcal{N}_{ii}V_i+\sigma_{i+1}\mathcal{N}_{i,i+1}V_{i+1}\\ \sigma_{i+1}^{-1}(v_{\Omega_{i+1}})_{\Gamma_i}-\sigma_i^{-1}(v_{\Omega_i})_{\Gamma_i}=\mathcal{D}_{i,i-1}V_{i-1}-2\mathcal{D}_{ii}V_i+\mathcal{D}_{i,i+1}V_{i+1}\\ \qquad -\sigma_i^{-1}\mathcal{S}_{i,i-1}p_{i-1}+(\sigma_i^{-1}+\sigma_{i+1}^{-1})\mathcal{S}_{ii}p_i-\sigma_{i+1}^{-1}\mathcal{S}_{i,i+1}p_{i+1}\end{cases}$$

$$\mathcal{S}_{ij}\psi(\boldsymbol{r})=\int_{\Gamma_j}G(\boldsymbol{r}-\boldsymbol{r}')\psi(\boldsymbol{r}')\mathrm{d}S(\boldsymbol{r}')$$

$$\mathcal{D}_{ij}\phi(\boldsymbol{r})=\mathrm{p.v.}\int_{\Gamma_j}\partial_{\boldsymbol{n}'}G(\boldsymbol{r}-\boldsymbol{r}')\phi(\boldsymbol{r}')\mathrm{d}S(\boldsymbol{r}')$$

$$\mathcal{D}^*_{ij}\psi(\boldsymbol{r})=\mathrm{p.v.}\int_{\Gamma_j}\partial_{\boldsymbol{n}}G(\boldsymbol{r}-\boldsymbol{r}')\psi(\boldsymbol{r}')\mathrm{d}S(\boldsymbol{r}')$$

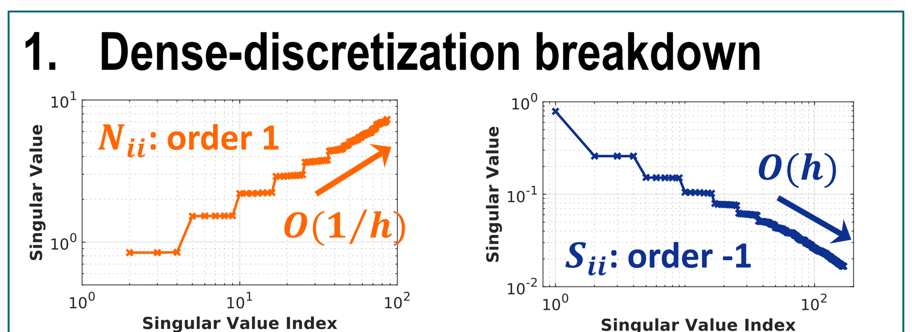
$$\mathcal{N}_{ij}\phi(\boldsymbol{r})=\mathrm{f.p.}\int_{\Gamma_j}\partial_{\boldsymbol{n}}\partial_{\boldsymbol{n}'}G(\boldsymbol{r}-\boldsymbol{r}')\phi(\boldsymbol{r}')\mathrm{d}S(\boldsymbol{r}')$$

- Its discretization by means of the standard **Boundary Element Method** (**BEM**) leads to a system of linear equations affected by two sources of **ill-conditioning**

$$\boldsymbol{Z}\begin{pmatrix}\boldsymbol{v}^{\mathrm{T}} & \boldsymbol{p}^{\mathrm{T}}\end{pmatrix}^{\mathrm{T}}=\begin{pmatrix}\boldsymbol{b}^{\mathrm{T}} & \boldsymbol{c}^{\mathrm{T}}\end{pmatrix}^{\mathrm{T}}$$

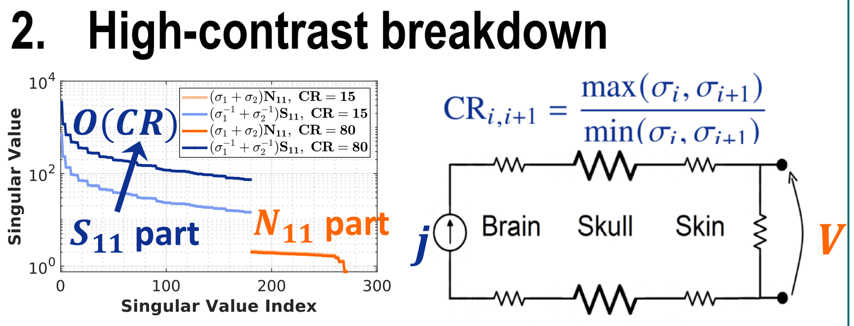
$$\boldsymbol{Z}=\begin{pmatrix}(\sigma_1+\sigma_2)\boldsymbol{N}_{11} & -\sigma_1\boldsymbol{N}_{12} & \boldsymbol{0} & -2\boldsymbol{D}^*_{11} & \boldsymbol{D}^*_{12}\\ -\sigma_1\boldsymbol{N}_{21} & (\sigma_2+\sigma_3)\boldsymbol{N}_{22} & -\sigma_2\boldsymbol{N}_{23} & \boldsymbol{D}^*_{21} & -2\boldsymbol{D}^*_{22}\\ \boldsymbol{0} & -\sigma_2\boldsymbol{N}_{32} & (\sigma_3)\boldsymbol{N}_{33} & \boldsymbol{0} & \boldsymbol{D}^*_{32}\\ -2\boldsymbol{D}_{11} & \boldsymbol{D}_{12} & \boldsymbol{0} & (\sigma_1^{-1}+\sigma_2^{-1})\boldsymbol{S}_{11} & -\boldsymbol{S}_{12}/\sigma_1\\ \boldsymbol{D}_{21} & -2\boldsymbol{D}_{22} & \boldsymbol{D}_{23} & -\boldsymbol{S}_{21}/\sigma_1 & (\sigma_2^{-1})\boldsymbol{S}_{22}\end{pmatrix}$$

1. Dense-discretization breakdown



$cond(\boldsymbol{Z})=O(1/h^2)$

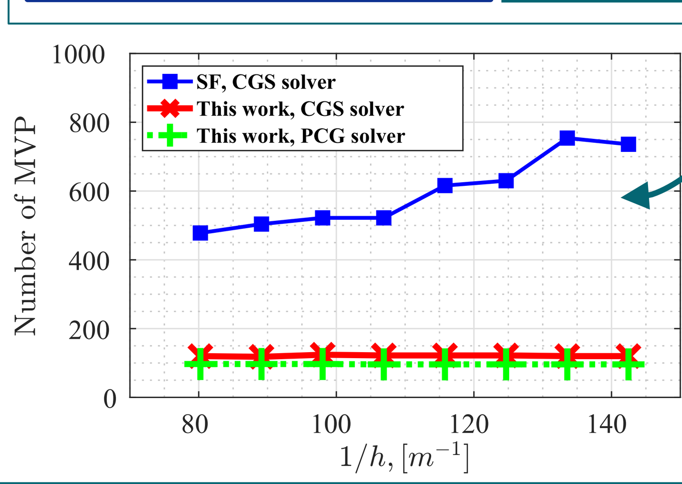
2. High-contrast breakdown



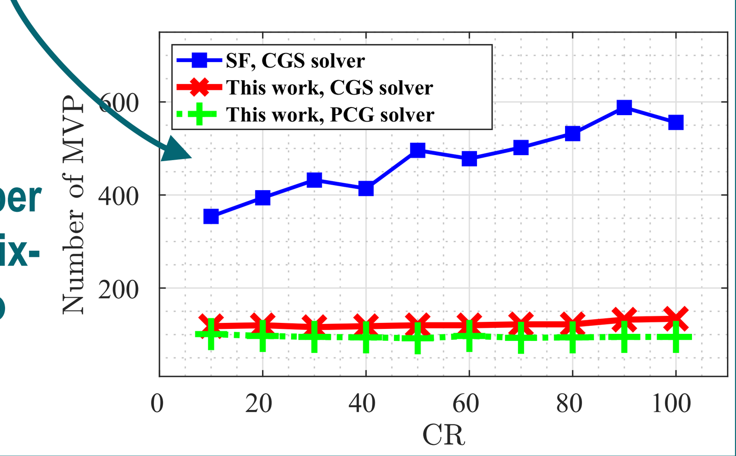
In the brain $CR\in[15,80]$

$cond(\boldsymbol{Z})=O(CR)$

$cond(Z)$, given by the ratio between max. and min. singular values of Z, is the key parameter for evaluating the stability of the problem



Increase of the number of iterations (or matrix-vector products) to solve the system iteratively.

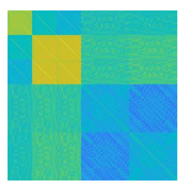


Submitted and published works

- Giunzioni V., Ortiz G. John E., Merlini A., Adrian S.B., Andriulli F.P., "A New Refinement-Free Preconditioner for the Symmetric Formulation in Electroencephalography", IEEE AP-S, URSI 2022, Denver, 2022
- Giunzioni V., Ortiz G. John E., Merlini A., Adrian S.B., Andriulli F.P., "On a Refinement-Free Calderon Preconditioner for the Symmetric Formulation of the EEG forward problem", Journal of Computational Physics, submitted in Sept. 2022

Novel contributions

- Our new formulation gives rise to a **symmetric**, **positive-definite** interaction matrix.
- Its construction is **refinement-free**: no need of dual functions defined on the barycentric refinement of the mesh.
- The key idea: $\mathcal{N}_{ii}\Delta_{\Gamma_i}^{-1}\mathcal{N}_{ii}$ and $\mathcal{S}_{ii}\Delta_{\Gamma_i}\mathcal{S}_{ii}$ are **second-kind** integral operators, of pseudo-differential **order 0**, hence they are well-conditioned.



$$\boldsymbol{Q}=\mathrm{diag}(q_{V,1}\boldsymbol{I}_{N_{V,1}},q_{V,2}\boldsymbol{I}_{N_{V,2}},\dots,q_{V,N}\boldsymbol{I}_{N_{V,N}},q_{C,1}\boldsymbol{I}_{N_{C,1}},q_{C,2}\boldsymbol{I}_{N_{C,2}},\dots,q_{C,N-1}\boldsymbol{I}_{N_{C,N-1}})$$

$$q_{V,i}=\max(\sigma_i,\sigma_{i+1})^{-1/2}\qquad q_{C,i}=\min(\sigma_i,\sigma_{i+1})^{1/2}$$

$$\boldsymbol{Z}_q=\boldsymbol{Q}\,\boldsymbol{Z}\,\boldsymbol{Q}$$

$$\begin{cases}\boldsymbol{M}\,\boldsymbol{Z}_q\,\boldsymbol{P}\,\boldsymbol{Z}_q\,\boldsymbol{M}\;\boldsymbol{y}=\boldsymbol{M}\,\boldsymbol{Z}_q\,\boldsymbol{P}\,\boldsymbol{Q}\begin{pmatrix}\boldsymbol{b}^{\mathrm{T}} & \boldsymbol{c}^{\mathrm{T}}\end{pmatrix}^{\mathrm{T}}\\ \begin{pmatrix}\boldsymbol{v}^{\mathrm{T}} & \boldsymbol{p}^{\mathrm{T}}\end{pmatrix}^{\mathrm{T}}=\boldsymbol{Q}\,\boldsymbol{M}\,\boldsymbol{y}\end{cases}$$

Our new **well-conditioned** symmetric formulation

$$\boldsymbol{M}=\mathrm{diag}\left(\boldsymbol{G}_{1,\pi\pi}^{-1/2},\boldsymbol{G}_{2,\pi\pi}^{-1/2},\dots,\boldsymbol{G}_{N,\pi\pi}^{-1/2},\boldsymbol{G}_{1,\lambda\lambda}^{-1/2},\boldsymbol{G}_{2,\lambda\lambda}^{-1/2},\dots,\boldsymbol{G}_{N-1,\lambda\lambda}^{-1/2}\right)$$

$$\boldsymbol{P}=\mathrm{diag}\left(\boldsymbol{\Delta}_1^+,\boldsymbol{\Delta}_2^+,\dots,\boldsymbol{\Delta}_N^+,\tilde{\boldsymbol{\Delta}}_{\pi,1},\tilde{\boldsymbol{\Delta}}_{\pi,2},\dots,\tilde{\boldsymbol{\Delta}}_{\pi,N-1}\right)$$

$$\tilde{\boldsymbol{\Delta}}_{\pi,i}=\boldsymbol{G}_{i,\tilde{\lambda}\pi}^{-1}\tilde{\boldsymbol{\Delta}}_i\boldsymbol{G}_{i,\pi\tilde{\lambda}}^{-1}$$

$$(\boldsymbol{G}_{i,fg})_{mn}=\left(f_{i,m}(\boldsymbol{r}),\mathcal{I}g_{i,n}(\boldsymbol{r})\right)_{L^2(\Gamma_i)}$$

$$\mathcal{N}_{ii}\Delta_{\Gamma_i}^{-1}\mathcal{N}_{ii}\qquad\mathcal{S}_{ii}\Delta_{\Gamma_i}\mathcal{S}_{ii}$$

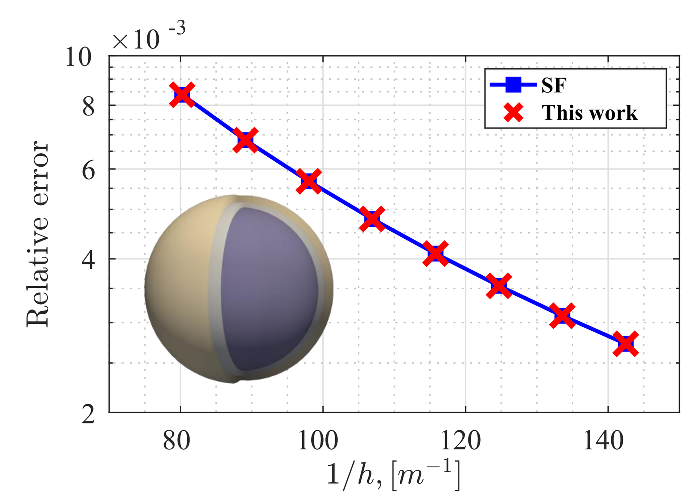
+1 -2 +1 -1 +2 -1

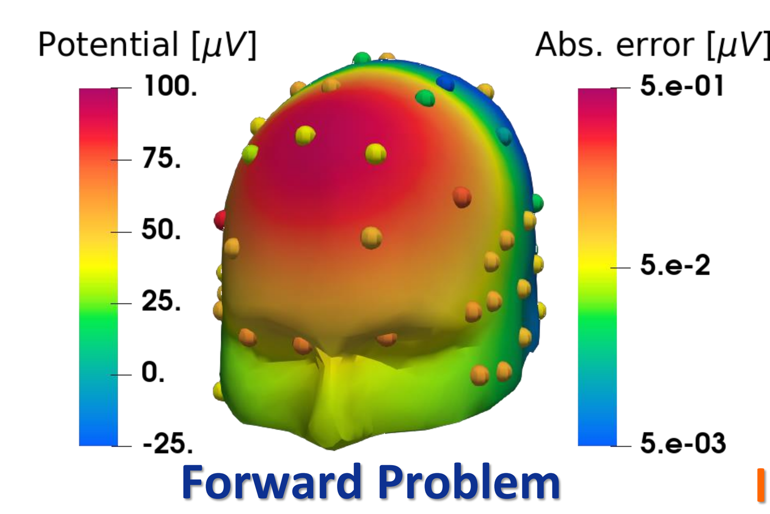
Adopted methodologies

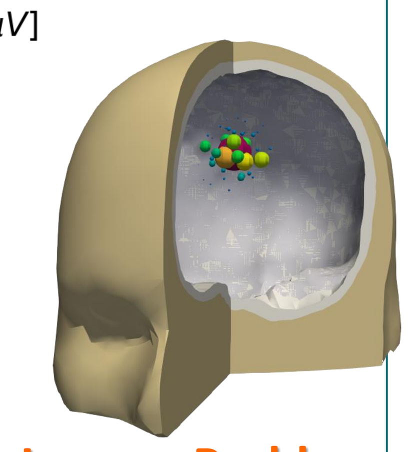
1. **Theoretical analysis**: the proposed preconditioning is based **on pseudo-differential operator theory**. Therefore, the **theoretical proof** of its good conditioning is based on the **spectral analysis** of each block, with the aim of determining its pseudo-differential order. Then, the analytic evaluation of the eigenvalues of the **principal part** of the formulation, independent of both mesh refinement and high contrast, ensures the **asymptotic boundness** of the condition number of the entire formulation, away from singularities.

$$\boldsymbol{Z}_q\,\boldsymbol{P}\,\boldsymbol{Z}_q=\begin{pmatrix}\alpha(\sigma)\underbrace{\boldsymbol{N}\boldsymbol{\Delta}^+\boldsymbol{N}}_{\text{order }0}+\beta(\sigma)\underbrace{\boldsymbol{D}^*\tilde{\boldsymbol{\Delta}}_\pi\boldsymbol{D}}_{\text{order }0} & \gamma(\sigma)\underbrace{\boldsymbol{N}\boldsymbol{\Delta}^+\boldsymbol{D}^*}_{\text{order }-2}+\delta(\sigma)\underbrace{\boldsymbol{D}^*\tilde{\boldsymbol{\Delta}}_\pi\boldsymbol{S}}_{\text{order }0}\\ \epsilon(\sigma)\underbrace{\boldsymbol{D}\boldsymbol{\Delta}^+\boldsymbol{N}}_{\text{order }-2}+\zeta(\sigma)\underbrace{\boldsymbol{S}\tilde{\boldsymbol{\Delta}}_\pi\boldsymbol{D}}_{\text{order }0} & \eta(\sigma)\underbrace{\boldsymbol{D}\boldsymbol{\Delta}^+\boldsymbol{D}^*}_{\text{order }-4}+\theta(\sigma)\underbrace{\boldsymbol{S}\tilde{\boldsymbol{\Delta}}_\pi\boldsymbol{S}}_{\text{order }0}\end{pmatrix}$$

2. **Numerical verification**: the stability of the formulation has also been shown **numerically**, on both canonical and realistic head models.







Future work

- **Acceleration** of the numerical solution of the preconditioned system by means of fast solvers, such as the **Adaptive Cross Approximation** (**ACA**).
- Extension of the formulation to include realistic features of the brain models, such as **anisotropy** and **neuronal fibers self-interactions**.

List of attended classes

- 01DPJRV – Lens antennas: Fundamentals and present applications (7/12/2021, 10 h.)
- 01DOBRV – Mathematical-physical theory of electromagnetism (11/7/2022, 15 h)
- 01SFVRV – Metamaterials: Theory and multiphysics applications (1/4/2022, 20 h)
- 01UIZRV – Microwave sensing and imaging for innovative applications in health and food industry (22/3/2022, 20 h)
- 02LWHRV – Communication (24/2/2022, 5 h)
- 01SHMRV – Entrepreneurial Finance (25/2/2022, 5 h)
- 01UNVRV – Navigating the hiring process: CV, tests, interview (14/2/2022, 2 h)
- 01UNYRV – Personal branding (14/2/2022, 1 h)
- 08IXTRV – Project management (24/2/2022, 5 h)
- 01RISRV – Public speaking (10/2/2022, 5h)
- 01SYBRV – Research integrity (21/2/2022, 5 h)
- 01SWQRV- Responsible research and innovation (22/2/2022, 5 h)
- 02RHORV – The new Internet Society (24/2/2022, 6 h)
- 01UNXRV – Thinking out of the box (10/2/2022, 1h)
- 01SWPRV – Time management (22/2/2022, 2h)